Deformation quantization with traces

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Abstract

In the present paper we prove a statement closely related to the cyclic formality conjecture [Sh]. In particular, we prove that for a constant volume form Ω and a Poisson bivector field π on $\mathbb{R}^d$ such that $\mathrm{div}_\Omega\,\pi = 0$, the Kontsevich star-product [K] with the harmonic angle function is cyclic, i.e. $\int_{\mathbb{R}^d} (f * g) \cdot h \cdot \Omega = \int_{\mathbb{R}^d} (g * h) \cdot f \cdot \Omega$ for any three functions f, g, h on $\mathbb{R}^d$ (for which the integrals make sense). We also prove a globalization of this theorem in the case of arbitrary Poisson manifolds and an arbitrary volume form, and prove a generalization of the Connes-Flato-Sternheimer conjecture [CFS] on closed star-products in the Poisson case.

1 Cyclic formality conjecture

We work with the algebra $A = C^\infty(M)$ of smooth functions on a smooth manifold M. One associates to the algebra A two differential graded Lie algebras: the Lie algebra $T^\bullet_{\mathrm{poly}}(M)$ of smooth polyvector fields on the manifold M (with zero differential and the Schouten-Nijenhuis bracket), and the polydifferential part $\mathcal{D}^\bullet_{\mathrm{poly}}(M)$ of the cohomological Hochschild complex of the algebra A, equipped with the Gerstenhaber bracket (see [K] for the definitions). We consider $T^\bullet_{\mathrm{poly}}(M)$ and $\mathcal{D}^\bullet_{\mathrm{poly}}(M)$ to be graded as Lie algebras, i.e. $T^i_{\mathrm{poly}}(M) = \{(i+1)\text{-polyvector fields}\}$ and $\mathcal{D}^i_{\mathrm{poly}}(M) \subset \mathrm{Hom}_{\mathbb{C}}\,(A^{\otimes(i+1)}, A)$.

The formality theorem of Maxim Kontsevich [K] states that $T^\bullet_{\mathrm{poly}}(M)$ and $\mathcal{D}^\bullet_{\mathrm{poly}}(M)$ are quasi-isomorphic as differential graded (dg) Lie algebras, i.e.

there exists a *dg* Lie algebra $\mathcal{M}^\bullet$ and the diagram

$$\begin{tikzcd} & \mathcal{M}^\bullet & \\ T^\bullet_{\text{poly}} \arrow[ur, "\phi_1"] & & \mathcal{D}^\bullet_{\text{poly}} \arrow[ul, "\phi_2"'] \end{tikzcd}$$

where ϕ_1 and ϕ_2 are maps of the Lie algebras and quasi-isomorphisms of the complexes.

In the case $M = \mathbb{R}^d$ an explicit L_∞-quasi-isomorphism $\mathcal{U} : T^\bullet_{\text{poly}}(\mathbb{R}^d) \to \mathcal{D}^\bullet_{\text{poly}}(\mathbb{R}^d)$ was constructed.

The cyclic formality conjecture relates to the formality theorem like the cyclic complex of an associative algebra A relates to its Hochschild complex. It turns out that the definition of the *cohomological* cyclic complex depends on an additional datum − a trace $\mathrm{Tr} : A \to \mathbb{C}$ on the algebra A. In the case $A = C^\infty(M)$ it depends on a volume form Ω on the manifold M.

Let us suppose that the volume form Ω is fixed.

Definition-lemma. (cyclic shift operator)
For any polydifferential Hochschild cochain $\psi : C^\infty(M)^{\otimes k} \to C^\infty(M)$ there exists a polydifferential Hochschild cochain $C(\psi) : C^\infty(M)^{\otimes k} \to C^\infty(M)$ such that for any $k+1$ functions $f_1, \ldots, f_{k+1}$ on M with compact support one has:

$$\int_{\mathbb{R}^d} \psi(f_1 \otimes \ldots \otimes f_k) \cdot f_{k+1} \cdot \Omega = (-1)^k \int_{\mathbb{R}^d} C(\psi)(f_2 \otimes \ldots \otimes f_{k+1}) \cdot f_1 \cdot \Omega\,. \quad (1)$$

Definition. (cohomological cyclic complex)

$$[\mathcal{D}^\bullet_{\text{poly}}(M)]_{\text{cycl}} = \{\psi \in \mathcal{D}^\bullet_{\text{poly}}(M) \mid C(\psi) = \psi\}\,.$$

Lemma. $[\mathcal{D}^\bullet_{\text{poly}}(M)]_{\text{cycl}}$ *is closed under the Hochschild differential and the Gerstenhaber bracket.*

See [Sh], Section 1.3.2 for a proof.

We have defined a cyclic analog of the *dg* Lie algebra $\mathcal{D}^\bullet_{\text{poly}}(M)$. The cyclic analog of $T^\bullet_{\text{poly}}(M)$ is defined as follows: it is $T^\bullet_{\text{poly}}(M) \otimes \mathbb{C}\,[u]$, $\deg u = 2$ with the $\mathbb{C}\,[u]$-linear bracket and the differential $d_{\text{div}}(\gamma \otimes u^k) = (\text{div}\,\gamma) \otimes u^{k+1}$. The

divergence operator $\mathrm{div} : T^\bullet_{\mathrm{poly}}(M) \to T^{\bullet-1}_{\mathrm{poly}}(M)$ is defined from the volume form Ω and the de Rham operator:

$$\mathrm{div} : T^k_{\mathrm{poly}} \xrightarrow{\Omega} \Omega^{d-k-1}(M) \xrightarrow{d_{\mathrm{DR}}} \Omega^{d-k}(M) \xrightarrow{\Omega} T^{k-1}_{\mathrm{poly}}(M)$$

(here $d = \dim M$).

We have to prove that $\{T^\bullet_{\mathrm{poly}} \otimes \mathbb{C}\,[u], d_{\mathrm{div}}\}$ is actually a *dg* Lie algebra, i.e. to prove that

$$\mathrm{div}\,[\gamma_1, \gamma_2] = [\mathrm{div}\,\gamma_1, \gamma_2] \pm [\gamma_1, \mathrm{div}\,\gamma_2]\,. \tag{2}$$

It follows from the fact that for any volume form Ω

$$\pm\,(\,\mathrm{div}\,(\gamma_1 \wedge \gamma_2) - (\mathrm{div}\,\gamma_1) \wedge \gamma_2 \pm \gamma_1 \wedge \mathrm{div}\,\gamma_2) = [\gamma_1, \gamma_2]\,. \tag{3}$$

Formula (2) can be obtained from (3) by the application of div to both sides and from the identity $\mathrm{div}^2 = 0$.

The cyclic formality is the following

Conjecture. *The dg Lie algebras*

$$\{T^\bullet_{\mathrm{poly}} \otimes \mathbb{C}\,[u], d_{\mathrm{div}}\} \quad \textit{and} \quad \{[\mathcal{D}^\bullet_{\mathrm{poly}}(M)]_{\mathrm{cycl}}, d_{\mathrm{Hoch}}\}$$

are quasi-isomorphic for any manifold M and volume form Ω.

This conjecture is due to M. Kontsevich (unpublished). In [Sh] there was constructed (conjecturally) an explicit L_∞-quasi-isomorphism in the case $M = \mathbb{R}^d$.

One can consider also $[T^\bullet_{\mathrm{poly}}(M)]_{\mathrm{div}} = \{\gamma \in T^\bullet_{\mathrm{poly}}(M) \mid \mathrm{div}\,\gamma = 0\}$ as a *dg* Lie algebra with zero differential. The main result of the present paper is the following.

Theorem. *Let Ω be a constant volume form on $\mathbb{R}^d$. Then the restriction of Kontsevich's L_∞-quasi-isomorphism $\mathcal{U} : T^\bullet_{\mathrm{poly}}(\mathbb{R}^d) \to \mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{R}^d)$, constructed from the angle function*

$$\varphi^h(z, w) = \frac{1}{2i} \log \left(\frac{(z - w)(z - \overline{w})}{(\overline{z} - w)(\overline{z} - \overline{w})} \right),$$

to the Lie subalgebra $[T^\bullet_{\mathrm{poly}}(\mathbb{R}^d)]_{\mathrm{div}}$, defines an L_∞-map $\mathcal{U} : [T^\bullet_{\mathrm{poly}}(\mathbb{R}^d)]_{\mathrm{div}} \to [\mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{R}^d)]_{\mathrm{cycl}}$.

In other words, the components $\mathcal{U}_k(\gamma_1\wedge\dots\wedge\gamma_k)$ of the Kontsevich L_∞-morphism are cyclic, if

$$\operatorname{div}\gamma_1=\dots=\operatorname{div}\gamma_k=0$$

(with respect to a constant volume form).

Corollary. *For a Poisson bivector field π and a constant volume form Ω on $\mathbb{R}^d$ such that $\operatorname{div}_\Omega\pi=0$, the Kontsevich star-product, constructed from π, is cyclic, i.e.*

$$\int_{\mathbb{R}^d}(f*g)\cdot h\cdot\Omega=\int_{\mathbb{R}^d}(g*h)\cdot f\cdot\Omega$$

for any three functions $f,g,h\in C^\infty(\mathbb{R}^d)$ with compact support.

Remark. Let us note that the complexes $[T^\bullet_{\mathrm{poly}}(M)]_{\mathrm{div}}$ and $[\mathcal{D}^\bullet_{\mathrm{poly}}(M)]_{\mathrm{cycl}}$ have different cohomology (see [Sh], Section 2), in particular, the L_∞-morphism

$$\mathcal{U}:[T^\bullet_{\mathrm{poly}}(\mathbb{R}^d)]_{\mathrm{div}}\to[\mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{R}^d)]_{\mathrm{cycl}}$$

is not a quasi-isomorphism.

We prove the theorem in the next section and globalize it to the case of an arbitrary manifold M with a volume form in Section 3. In particular, we prove that the statement of the corollary holds for any volume form on $\mathbb{R}^d$ but with a star-product which does not in general coincide with Kontsevich's one.

2 Geometry of the cyclic formality conjecture

2.1

In this section we recall Kontsevich's construction [K] of L_∞-morphism of formality, but in a slightly different form. In fact, we replace the 2-dimensional group $G^{(2)}=\{z\mapsto az+b\,;\ a,b\in\mathbb{R}\,,\ a>0\}$ by the whole group $\mathrm{PSL}_2(\mathbb{R})=\left\{z\mapsto\frac{az+b}{cz+d}\,;\ a,b,c,d\in\mathbb{R}\,,\ \det\begin{pmatrix}a&c\\b&d\end{pmatrix}=1\right\}$. Actually, the group $G^{(2)}$ is exactly the subgroup in $\mathrm{PSL}_2(\mathbb{R})$, preserving the point ∞.

We consider the disk $D^2 = \{z \mid |z| \leq 1\}$ instead of the upper half-plane $\mathcal{H}$ in [K], and we identify it with $\mathcal{H} \cup \{\infty\}$ by stereographic projection. In particular the group $\mathrm{PSL}_2(\mathbb{R})$ of holomorphic transformations of $\mathbb{C}P^1$ preserving the upper half-plane acts on D^2. Now an *admissible graph* is the same as in [K], but the vertices $\{1, 2, \dots, n\}$ of the first type are placed in the interior of the disk D^2, the vertices $\{\overline{1}, \overline{2}, \dots, \overline{m}\}$ of the second type are placed on the boundary $S^1 = \partial D^2$, and $2n + m - 3 \geq 0$. In particular, there are no simple loops, and for every vertex $k \in \{1, 2, \dots, n\}$ of the first type, the set of edges $\mathrm{Star}\,(k) = \{(v_1, v_2) \in E_\Gamma \mid v_1 = k\}$, starting from k, is labelled by symbols $(\ell_k^1, \dots, \ell_k^{\#\mathrm{Star}(k)})$.

To each such graph Γ with $2n + m - 3 + \ell$ edges, we attach a linear map

$$\widetilde{\mathcal{U}}_\Gamma : \otimes^n T^\bullet_{\mathrm{poly}}(\mathbb{R}^d) \to \mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{R}^d)\,[2 + \ell - n]\,,$$

exactly as in [K], Section 6.3.

Now we are going to define the weight W_Γ for a graph Γ with n vertices of the first type, m vertices of the second type, and $2n + m - 3$ edges.

We consider configuration spaces $\widetilde{\mathrm{Conf}}_{n,m} = \{(p_1, \dots, p_n\,;\; q_1, \dots, q_m) \mid p_i \in \mathrm{Int}\, D^2\,,\; q_j \in S^1 = \partial D^2\,,\; p_{i_1} \neq p_{i_2} \text{ for } i_1 \neq i_2 \text{ and } q_{j_1} \neq q_{j_2} \text{ for } j_1 \neq j_2\}$. We suppose $2n + m \geq 3$. Then the group $\mathrm{PSL}_2(\mathbb{R})$ acts freely on $\widetilde{\mathrm{Conf}}_{n,m}$. We set $D_{n,m} = \widetilde{\mathrm{Conf}}_{n,m}/\mathrm{PSL}_2(\mathbb{R})$; $\dim D_{n,m} = 2n+m-3$. One can construct a compactification $\overline{D}_{n,m}$ of the space $D_{n,m}$ exactly as in [K], Section 5, but we will not use it.

Now to each graph Γ as above we attach a differential form of top degree on the space $D_{n,m}$. Let $\alpha_1, \dots, \alpha_m$ be (positive) real numbers, and let $p, q \in D^2$.

Definition. Let $\xi_1, \dots, \xi_m \in S^1$. Let $\phi_i(p, q)$, $i = 1, \dots, m$, be the angle between the geodesic in the Poincaré metric on D^2, connecting p and q, and the geodesic, connecting p and a point ξ_i on S^1 (where "sits" the i-th vertex of the second type), see Figure 1. It is defined modulo 2π. We set $\phi_{\alpha_1,\dots,\alpha_m}(p, q) = \sum_{k=1}^{m} \alpha_k\, \phi_k(p, q)$. The function $\phi_{\alpha_1,\dots,\alpha_m}(p, q)$ depends on the points $p, q \in D^2$ and on the points $\xi_1, \dots, \xi_m \in S^1$. The differential $d\phi_{\alpha_1,\dots,\alpha_m}(p, q)$ is a well-defined 1-form on $\widetilde{\mathrm{Conf}}_{2,m}$. It is $\mathrm{PSL}_2(\mathbb{R})$-invariant, where $\mathrm{PSL}_2(\mathbb{R})$ acts simultaneously on p, q and on $\xi_1, \dots, \xi_m$.

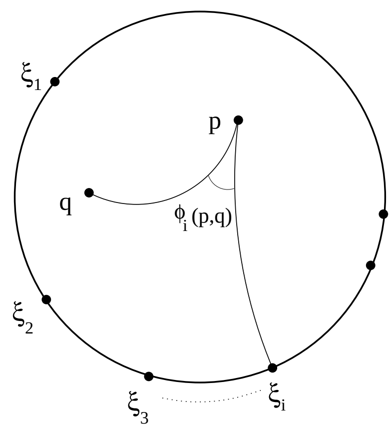

Figure 1

Example. If $(\alpha_1, \ldots, \alpha_m) = (0, \ldots, 0, 1)$ and $\xi_m = \infty$, then

$$\phi_{\alpha_1,\ldots,\alpha_m}(p,q) = \frac{1}{2i}\log\left(\frac{(p-q)(p-\overline{q})}{(\overline{p}-q)(\overline{p}-\overline{q})}\right),$$

as in [K].

Now, if the graph Γ is "placed" on the disk D^2, one associates to each edge e the 1-form $d\phi_{e;\alpha_1,\ldots,\alpha_m}$ on $\widetilde{\mathrm{Conf}}_{n,m}$. It is the pull-back of $d\phi_{\alpha_1,\ldots,\alpha_m}$ by the map sending the two points connected by the edge to p,and q, and q_j to ξ_j, $j = 1, \ldots, m$. This 1-form induces a 1-form on the space $D_{n,m}$.

Definition. (weight W_Γ)

$$W_\Gamma^{(\alpha_1,\ldots,\alpha_m)} = \prod_{k=1}^{n} \frac{1}{(\#\,\mathrm{Star}\,(k))!} \cdot \frac{1}{(2\pi)^{2n+m-2}} \cdot \int_{D^+_{n,m}} \bigwedge_{e\in E_\Gamma} d\,\phi_{e;\alpha_1,\ldots,\alpha_m}\,.$$

Here $D^+_{n,m}$ is the connected component consisting of configurations for which the points q_j, $j = 1, \ldots, m$, are cyclically ordered counterclockwise.

To each map $\mathcal{U}_\Gamma : \otimes^n T^\bullet_{\mathrm{poly}} \to \mathcal{D}^\bullet_{\mathrm{poly}}[2-n]$ one associates the corresponding skew-symmetric map $\widetilde{\mathcal{U}}_T : \bigwedge^n T^\bullet_{\mathrm{poly}} \to \mathcal{D}^\bullet_{\mathrm{poly}}[2-n]$. We define

$$\widetilde{\mathcal{U}}_n^{\alpha_1,\ldots,\alpha_m} = \sum_{m\geq 0} \sum_{\Gamma\in G_{n,m}} W_\Gamma^{\alpha_1,\ldots,\alpha_m} \times \widetilde{\mathcal{U}}_\Gamma$$

where $G_{n,m}$ is the set of admissible graphs with n vertices of the first type, m vertices of the second type, and $2n + m - 3$ edges.

Lemma. *In the case* $(\alpha_1,\dots,\alpha_m)=(0,0,\dots,0,1)$

$$\widetilde{\mathcal{U}}_n^{\alpha_1,\dots,\alpha_m}(f_1\otimes\cdots\otimes f_m)=\mathcal{U}_n(f_1\otimes\cdots\otimes f_{m-1})\cdot f_m\,,$$

where $\mathcal{U}_n$ *is the Taylor component of Kontsevich's* L_∞*-map ([K], Section 6).*

Proof. Using $\mathrm{PSL}_2(\mathbb{R})$-invariance, one can assume that f_m "sits" in the point $\{\infty\}$, and then the configuration is $G^{(2)}$-invariant. By definition of $\phi_{e;\alpha_1,\dots,\alpha_m}$, if the graph Γ contains an edge ending at $\{\infty\}$, the weight $W_\Gamma^{\alpha_1,\dots,\alpha_m}$ vanishes (the corresponding angle between two geodesics is zero). $\square$

2.2

Theorem. *Let* Ω *be a constant volume form on* $\mathbb{R}^d$*, and let* $\gamma_1,\dots,\gamma_n\in[T^\bullet_{\mathrm{poly}}(\mathbb{R}^d)]_{\mathrm{div}}$*. Then the integral* $\int_{\mathbb{R}^d}\widetilde{\mathcal{U}}_n^{\alpha_1,\dots,\alpha_m}(\gamma_1\wedge\cdots\wedge\gamma_n)(f_1\otimes\cdots\otimes f_m)\cdot\Omega$ *depends only on the sum* $\alpha_1+\cdots+\alpha_m$*.*

The theorem in Section 1 follows from this theorem and Lemma 2.1, when we set $(\alpha_1,\dots,\alpha_m)=(0,0,\dots,0,1)$ and $(\alpha'_1,\dots,\alpha'_m)=(1,0,\dots,0)$.

Proof of the theorem. We consider the case $(\alpha_1,\dots,\alpha_m)=(0,0,\dots,0,1)$ and $(\alpha'_1,\dots,\alpha'_m)=(1,0,\dots,0)$; the general case is analogous.

Key-lemma.

$$d\,\phi_{\alpha_1,\dots,\alpha_m}(p,q)=d\,\phi_{\alpha'_1,\dots,\alpha'_m}(p,q)+d\Phi_{\{\alpha\},\{\alpha'\}}(p)$$

where the 1-form $d\Phi$ *does not depend on* q*.*

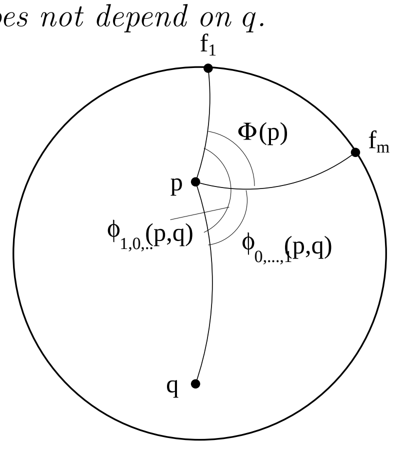

Figure 2

Proof. It follows directly from the additivity of the angle function, see Figure 2. □

We proceed to prove Theorem 2.2. The weight $W_\Gamma^{\alpha_1,\dots,\alpha_m}$ is the integral of the wedge product $\bigwedge_{e\in E_\Gamma} d\phi_{e;\alpha_1,\dots,\alpha_m}$. Now $d\phi_{e;\alpha_1',\dots,\alpha_m'} = d\phi_{e;\alpha_1,\dots,\alpha_m} + d\Phi_e(p)$. Then, by the skew-symmetry,

$$\bigwedge_{e\in \mathrm{Star}(k)} d\phi_{e;\alpha_1',\dots,\alpha_m'} = \tag{4}$$

$$\sum_{e\in\mathrm{Star}(k)} \left(\pm \bigwedge_{\overline{e}\in \mathrm{Star}(k)\backslash e} d\phi_{\overline{e};\alpha_1,\dots,\alpha_m}\right) \wedge d\Phi_e + \bigwedge_{e\in\mathrm{Star}(k)} d\phi_{e;\alpha_1,\dots,\alpha_m}\,.$$

Let us denote by ω_e the form

$$\left(\bigwedge_{\overline{e}\in \mathrm{Star}(k)\backslash e} d\phi_{\overline{e};\alpha_1,\dots,\alpha_m}\right) \wedge d\Phi_e \wedge \bigwedge_{\widetilde{e}\notin\mathrm{Star}(k)} d\phi_{\widetilde{e};\alpha_1,\dots,\alpha_m}$$

$(e\in\mathrm{Star}(k))$.

If $\Gamma' = (\Gamma\backslash\{e\})\coprod\{e'\}$ where the edges e and e' have the same start-point, then

$$\int_{D_{n,m}} \omega_e^\Gamma = \int_{D_{n,m}} \omega_{e'}^{\Gamma'}\,. \tag{5}$$

Now it is sufficiently to prove the following

Lemma. *Let $\overline{\Gamma}$ be a graph with n vertices of the first type, m vertices of the second type, and $2n+m-4$ edges; and let $\gamma_1,\dots,\gamma_n$ be arbitrary polyvector fields. Let $G^i_{\overline{\Gamma}}$ be the set of all the graphs, obtained from the graph $\overline{\Gamma}$ by addition of any edge e such that the start-point of e is $\{i\}$. Then*

$$\int_{\mathbb{R}^d} \sum_{\Gamma\in G^i_{\overline{\Gamma}}} \mathcal{U}_\Gamma(\gamma_1\wedge\cdots\wedge\gamma_n)(f_1\otimes\cdots\otimes f_m)\cdot\Omega \tag{6}$$

$$= \pm\int_{\mathbb{R}^d} \mathcal{U}_{\overline{\Gamma}}(\gamma_1\wedge\cdots\wedge \mathrm{div}\gamma_i\wedge\cdots\wedge\gamma_n)(f_1\otimes\cdots\otimes f_m)\cdot\Omega\,.$$

This is a standard result. The simplest version of it is the following: let ξ be a vector field on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \xi(f)\cdot\Omega = -\int \mathrm{div}\,(\xi)\cdot f\cdot\Omega\,.$$

Indeed, let us suppose that $\Omega = dx_1\wedge\cdots\wedge dx_d$, then, if $\xi = \sum a_i(x_1,\ldots,x_d)\,\frac{\partial}{\partial x_i}$, one has:

$$\int_{\mathbb{R}^d}\xi(f) = \int\sum a_i\cdot\partial_i(f) = \int\sum\partial_i(a_i\cdot f) - \int\sum\partial_i(a_i)\cdot f\,.$$

The first summand is equal to 0, and the second summand is $-\int(\operatorname{div}\xi)\cdot f$.

Now Theorem 2.2 follows from the last lemma and formula (5).

Theorem 2.2 is proven, and Theorem 1 follows now from Theorem 2.2 and Lemma 2.1. □

Remark. For a general volume form the proof fails, because (6) is not true.

Remark. We have proved the theorem for very special choice of angle function (in the sense of [K], Section 6.2), namely, we use the harmonic angle function, which also appears in the QFT approach to the formality theorem [CF]. It seems that Theorem 1 is not true for any other choice.

Remark. The cyclicity of the star product may be heuristically understood, for $M = \mathbb{R}^d$ with volume form $\Omega = dx_1\wedge\cdots\wedge dx_d$, in "physical" terms as follows: in [CF] the Kontsevich star product was described as the Feynman perturbation expansion of a path integral formula $f * g(x) = \int_{X(p_3)=x}\exp(\frac{i}{\hbar}S(\hat X))f(X(p_1))g(X(p_2))d\hat X$, for a certain action functional S on the space of bundle homomorphisms $\hat X : TD \to T^*M$, with base map $X : D \to M$, from the tangent bundle of an oriented disk D to the cotangent bundle of M. The points p_1, p_2, p_3 are any three cyclically ordered points on the boundary of D. One may more generally consider for any three functions $f, g, h \in C^\infty(M)$ the correlation function

$$\langle f, g, h\rangle = \int e^{\frac{i}{\hbar}S(\hat X)}f(X(p_1))g(X(p_2))h(X(p_3))d\hat X$$

which looks invariant under cyclic permutations of f, g, h, since the action is invariant under orientation preserving diffeomorphisms of D. Moreover, if $M = \mathbb{R}^d$, we may write this integral as the integral over the maps with $x = \sum\alpha_i X(p_i)$ fixed, and then over the position of the "center of mass" x. Naively, this is independent of α_i with $\sum\alpha_i = 1$. In particular, with $\alpha = (0,0,1)$ and $(1,0,0)$, we obtain

$$\int_{\mathbb{R}^d} f * g\cdot h\,dx_1\wedge\cdots\wedge dx_d = \int_{\mathbb{R}^d} g * h\cdot f\,dx_1\wedge\cdots\wedge dx_d.$$

However there is an anomaly, meaning that the symmetry under diffeomorphisms of the disk is not a symmetry of the path integral. Technically this follows from the fact that the regularization of amplitudes of Feynman diagrams involving tadpoles (edges with both ends at the same vertex) cannot be chosen in an invariant way, see [CF]. But the tadpoles correspond to bidifferential operators involving the divergence of the Poisson bivector field. Therefore the anomalous terms vanish for divergence free Poisson bivector fields and the above argument applies.

3 Globalization

3.1

Here we prove the following

Theorem. *For any smooth manifold M and any volume form Ω on M, there exists an L_∞-morphism*

$$\mathcal{U} : [T^\bullet_{\mathrm{poly}}(M)]_{\mathrm{div}} \to [\mathcal{D}^\bullet_{\mathrm{poly}}(M)]_{\mathrm{cycl}}\,,$$

such that its first Taylor component, $\mathcal{U}_1$, coincides with the Hochschild-Kostant-Rosenberg map.

The proof follows basically the same line as the proof of the globalization of the formality theorem in [K], Section 7. Let us note that in our case the L_∞-morphism $\mathcal{U} : [T^\bullet_{\mathrm{poly}}(M)]_{\mathrm{div}} \to [\mathcal{D}^\bullet_{\mathrm{poly}}(M)]_{\mathrm{cycl}}$ is *not* an L_∞-quasi-isomorphism.

For any d-dimensional manifold M, there exists an infinite-dimensional manifold M^{coor}, the manifold of formal coordinate systems on M. The main property of the manifold M^{coor} is that there exists a W_d-valued 1-form ω on M^{coor}, satisfying the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega,\omega] = 0$ (here $W_d = \mathrm{Vect}\,(\mathbb{R}^d_{\mathrm{formal}})$ is the Lie algebra of formal vector fields on $\mathbb{R}^d$). In other notations, there exists a map $T[1]\,M^{\mathrm{coor}} \to W_d[1]$, which is a map of Q-manifolds (we refer to [K] for basic definitions on Q-manifolds). We need to modify this construction in the case when the manifold M is equipped with a volume form Ω.

We define an infinite-dimensional manifold M^{coor}_Ω for any d-dimensional manifold M with volume form Ω, as follows. A point of the manifold M^{coor}_Ω is a map $\phi : \mathbb{R}^d_{\mathrm{formal}} \to M$, $\phi(0) = x \in M$ such that $\phi^*\,\Omega = dx_1 \wedge \cdots \wedge dx_d$.

Let us denote by $[W_d]_{\mathrm{div}}$ the Lie algebra of formal vector fields on $\mathbb{R}^d$ with zero divergence (with respect to the standard volume form $dx_1 \wedge \cdots \wedge dx_d$ on $\mathbb{R}^d$). More explicitly, $[W_d]_{\mathrm{div}} = \{\sum a_i(x_1, \ldots, x_n)\, \partial_i \mid \sum \partial_i\, a_i = 0\}$.

Remark. The volume form Ω on M is constant in the coordinates in a neighbourhood of the point x obtained through the map ϕ from the affine coordinates on $\mathbb{R}^d_{\mathrm{formal}}$.

There exists a $[W_d]_{\mathrm{div}}$-valued 1-form ω_{div} on M^{coor}_Ω, which satisfies the Maurer-Cartan equation $d\,\omega_{\mathrm{div}} + \frac{1}{2}\,[\omega_{\mathrm{div}}, \omega_{\mathrm{div}}] = 0$. In the other terms, there exists a map of Q-manifolds $T[1]\, M^{\mathrm{coor}}_\Omega \to [W_d]_{\mathrm{div}}\,[1]$. The Lie algebra $sl_d = \{\sum a_{ij}\, x_i\, \partial_j \mid a_{ij} \in \mathbb{C}\,,\ tr(a_{ij}) = 0\} \subset [W_d]_{\mathrm{div}}$ acts on M^{coor}_Ω (as well as the whole Lie algebra $[W_d]_{\mathrm{div}}$), and this action can be integrated to an action of the group SL_d.

Lemma. *The fibers of the natural bundle $M^{\mathrm{coor}}_\Omega/\mathrm{SL}_d \to M$ are contractible.*

Proof. Let $\phi : \mathbb{R}^d_{\mathrm{formal}} \to \mathbb{R}^d_{\mathrm{formal}}$, $\phi(0) = 0$, be a formal diffeomorphism, preserving the volume form $\Omega = dx_1 \wedge \cdots \wedge dx_d$. Then the map $\phi_\hbar : \mathbb{R}^d_{\mathrm{formal}} \to \mathbb{R}^d_{\mathrm{formal}}$,

$$\phi_\hbar(x_1, \ldots, x_d) = \frac{\phi(\hbar\, x_1, \ldots, \hbar\, x_d)}{\hbar}\,,\ \hbar \neq 0\,,$$

also preserves the volume form, and so does its limit when $\hbar \to 0$. It is clear that this limit is a linear map $\phi_0 : \mathbb{R}^d_{\mathrm{formal}} \to \mathbb{R}^d_{\mathrm{formal}}$, and, therefore, $\phi_0 \in \mathrm{SL}_d$. We have constructed a retraction of a fiber of the bundle $M^{\mathrm{coor}}_\Omega \to M$ on the space SL_d. $\square$

We use this lemma and the following properties of the L_∞-morphism $\mathcal{U} : [T^\bullet_{\mathrm{poly}}(\mathbb{R}^d)]_{\mathrm{div}} \to [\mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{R}^d)]_{\mathrm{cycl}}$:

P1) $\mathcal{U}$ can be defined for $\mathbb{R}^d_{\mathrm{formal}}$ as well;

P2) for any $\xi \in [W_d]_{\mathrm{div}}$ we have

$$\mathcal{U}_1(m_T(\xi)) = m_D(\xi)$$

(here $m_T : [W_d]_{\mathrm{div}} \to [T^\bullet_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})]_{\mathrm{div}}$ and $m_D : [W_d]_{\mathrm{div}} \to [\mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})]_{\mathrm{cycl}}$ are the canonical maps);

P3) $\mathcal{U}$ is SL_d-equivariant;

P4) for any $k \geq 2$, $\xi_1, \ldots, \xi_k \in [W_d]_{\mathrm{div}}$ one has

$$\mathcal{U}_k(m_T(\xi_1) \otimes \cdots \otimes m_T(\xi_k)) = 0\,;$$

P5) for any $k \geq 2$, $\xi \in sl_d \subset [W_d]_{\mathrm{div}}$ and for any $\eta_2, \ldots, \eta_k \in T^\bullet_{\mathrm{poly}}(\mathbb{R}^d_{\mathrm{formal}})$ one has:

$$\mathcal{U}_k(m_T(\xi) \otimes \eta_2 \otimes \cdots \otimes \eta_k) = 0\,.$$

The properties P1)–P5) are cited from [K], Section 7, where they were proven for the (a bit stronger) case of the L_∞-map of formality $\mathcal{U} : T^\bullet_{\mathrm{poly}}(\mathbb{R}^d) \to \mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{R}^d)$, we just replace the Lie algebra W_d by $[W_d]_{\mathrm{div}}$ and its subalgebra $gl_d \subset W_d$ by the subalgebra $sl_d \subset [W_d]_{\mathrm{div}}$. According to Remark 3.1, for the globalization it is sufficient to know the local result only for a constant volume form Ω.

The theorem can be deduced from the properties P1)-P5) exactly in the same way as it is done in the case of Formality in [K], Section 7.

3.2 Consequences

Here we consider some consequences of Theorem 3.1.

Corollary 1. *Let M be a Poisson manifold (with the bivector field π, and let Ω be any volume form Ω on M such that* $\mathrm{div}_\Omega\, \pi = 0$*. Then there exists a star-product on $C^\infty(M)$ such that for any three functions f, g, h with compact support one has:*

$$\int_M (f * g) \cdot h \cdot \Omega = \int_M (g * h) \cdot f \cdot \Omega\,.$$

Proof. We apply to the L_∞-morphism $\mathcal{U} : [T^\bullet_{\mathrm{poly}}(M)]_{\mathrm{div}} \to [\mathcal{D}^\bullet_{\mathrm{poly}}(M)]_{\mathrm{cycl}}$, constructed above, the following general statement. For an L_∞-morphism $\mathcal{F} : \mathcal{G}^\bullet_1 \to \mathcal{G}^\bullet_2$ between two dg Lie algebras and a solution π of the Maurer-Cartan equation in $(\mathcal{G}^\bullet_1)^1$, the formula

$$\mathcal{F}(\pi) = \mathcal{F}_1(\pi) + \frac{1}{2}\,\mathcal{F}_2(\pi, \pi) + \frac{1}{6}\,\mathcal{F}_3(\pi, \pi, \pi) + \cdots$$

defines a solution of the Maurer-Cartan equation in $(\mathcal{G}^\bullet_2)^1$. We apply this construction to the solution $\hbar\pi$ of the Maurer-Cartan equation in $[T^\bullet_{\mathrm{poly}}(M)]_{\mathrm{div}}$. $\square$

To deduce the Connes-Flato-Sternheimer conjecture from Corollary 1, one needs to prove that $1 * f = f * 1 = f$ for any function f. Locally it is true (for the L_∞-morphism $\mathcal{U} : [T^\bullet_{\rm poly}(\mathbb{R}^d)]_{\rm div} \to [\mathcal{D}^\bullet_{\rm poly}(\mathbb{R}^d)]_{\rm cycl}$, constructed in Section 2), therefore it is true also globally. So, we have proved

Corollary 2. (generalized Connes-Flato-Sternheimer conjecture) *For the star-product of Corollary 1 one has:*

$$\int_M (f * g) \cdot \Omega = \int_M f \cdot g \cdot \Omega\,.$$

Proof. Put $h = 1$ and use that $g * 1 = g$. $\square$

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